JOURNAL OF COMPUTATIONAL PHYSICS 1, 1-20 (1966)

An Analysis of the Errors Caused by Using Artificial Viscosity Terms to Represent Steady-State Shock Waves

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ABSTRACT

In calculating hydrodynamic flows by finite difference methods it is convenient to introduce artificial viscosity terms to smear out shocks, rather than to allow discontinuities in the physical variables. However, when these shocks cross a change in mesh or material, errors occur because of the transient nature of the shock. The object of this paper is to investigate the nature of these errors and to attempt to eliminate them. At a material interface it is found that the errors can be considerably reduced either by a suitable choice of mesh for the second material or by modifying the definition of the artificial viscosity term.

I. INTRODUCTION

When calculating hydrodynamic flows by finite difference methods it is often more convenient to introduce an artificial viscosity term to smear shocks out over a finite distance than to introduce discontinuities in the physical variables [1]. While artificial viscosity terms can be adjusted to give the correct entropy rise when representing steady shock waves, there will be an error when representing transient behavior, i.e., when shocks cross interfaces or when the finite difference mesh varies [2]. To calculate hydrodynamic flows accurately when there is only one space variable it is always possible to reduce these errors by increasing the number of meshes. For two-dimensional flows, however, the number of meshes is strictly limited by computer time and storage. Any effort to increase the accuracy of two-dimensional problems must come

through an understanding of the one-dimensional case. The object of this paper is to investigate and attempt to correct the errors that arise when a steady-state, one-dimensional shock represented by this artificial viscosity term crosses a change in mesh or material or both. These shocks are referred to as "viscous shocks," although this is purely a numerical representation and not a physical reality.

II. The Basic Equations

In a coordinate system moving with the steady shock speed U the system is stationary and the one-dimensional hydrodynamic equations may be written in integrated form as

$$\varrho w = \text{constant} = m, \tag{1}$$

$$p + q + mw = \text{constant}, \tag{2}$$

$$m(e + \tfrac{1}{2} w^2) + (p + q)w = \text{constant}, \tag{3}$$

where w is the velocity relative to the moving coordinate system, and p, e, ϱ, and q represent the pressure, internal energy, density, and the artificial viscosity term, respectively. The latter may take many forms, although for the present we will confine our attentions to the well-known von-Neumann q [1] defined by

$$q_1 = \begin{cases} L^2 \dfrac{\varrho_0^2}{\varrho} \left(\dfrac{\partial u}{\partial x}\right)^2 & \dfrac{\partial u}{\partial x} < 0, \\ 0 & \dfrac{\partial u}{\partial x} > 0, \end{cases} \tag{4}$$

where ϱ_0 is a reference density for the materiale, x is an Eulerian distance coordinate, and L is a constant having the dimensions of length. For finite difference methods we take $L = b\ \Delta x$, where b is a nondimensional constant and Δx is the mesh size.

Throughout this paper we shall use an equation of state of the so-called water type

$$e = \frac{(p + B)}{(n - 1)\varrho}, \tag{5}$$

where B and n are constants for any given material. For aluminium and uranium the values for B and n which give the best fit to experimental

data over the range 0–1 Mb are 1.067, 1.667, and 1.3325, 1.1, respectively. Eliminating p, w, and e from Eqs. (1) to (3) and (5) gives

$$q = \frac{m^2}{\varrho_1 y}\left(\frac{n+1}{2}\right)(1-y)(y-\lambda), \tag{6}$$

where ϱ_1 and ϱ_2 are the constant densities far shead and behind the shock, $y = \varrho_1/\varrho$ and $\lambda = \varrho_1/\varrho_2$. Since the derivation of Eq. (6) is independent of the form of q assumed in Eq. (4), we can see from Eq. (6) that the maximum value of q for a steady shock is

$$q_{\max} = \frac{m^2}{\varrho_1}\left(\frac{n+1}{2}\right)(1-\sqrt{\lambda})^2, \tag{7}$$

which is independent of the form of q and will be the same for other forms.

Substituting (4) into (6) and integrating gives

$$y = \frac{1+\lambda}{2} + \frac{1-\lambda}{2}\cos\frac{2}{\lambda+1}\left\{[(1-y)(y-\lambda)]^{1/2} + \left(\frac{n+1}{2}\right)^{1/2}\left(\frac{x-x_0}{L}\right)\right\}, \tag{8}$$

where x_0 is a reference point and we have taken $\varrho_0 = \varrho_1$. With this form of q the shock is spread out over a distance

$$X = \frac{\lambda+1}{2}\pi L\left(\frac{2}{n+1}\right)^{1/2}, \tag{9}$$

and the time for a complete wave to pass a point is

$$T = \frac{\lambda+1}{2}\frac{\pi L}{U}\left(\frac{2}{n+1}\right)^{1/2}; \tag{10}$$

i.e., T is the time taken for the material at any point to change from its initial to its final state through the passage of the shock wave.

III. The Density Error at an Interface

The velocity of a shock varies from one material to another, and so to a lesser degree does the shock width; therefore, the time taken for

a shock to pass a given point will differ from material to material. Hence, the shock transmitted across an interface will have, in general, the wrong initial width and speed producing wrong values of $\partial u/\partial x$. The same is true of the reflected shock which will have different properties from that of the incident shock.

For an ideal gas, i.e., one whose temperature T satisfies the equation

$$p\mathrm{v} = RT,$$

where R is the universal gas constant and v is the specific volume, the entropy change may be calculated from

$$\frac{1}{R}\frac{dS}{d\mathrm{v}} = -q/p\mathrm{v}, \tag{12}$$

where S is the entropy; unless q has the correct form the material will not receive the correct entropy change. If a mesh is at a different entropy from its neighbor, then for equal pressures it must have a different density, and this difference will always exist: the pressure differences of course will disperse with sonic velocities. Since the entropy error may be positive or negative the density error may also be positive or negative.

In a steady state a shock will take the same time to pass any particular point, although this time will vary for different meshes, materials, and shock strengths. Now when a shock crosses a change in mesh or material, it is while it is changing from one steady state to another that the material receives the wrong entropy increment: this error should be considerably reduced by making the time T for the shock to pass a fixed point the same for both sides of the interface. Writing $L = b\,\Delta x$ and matching T for the incident and transmitted shocks, we have

$$\pi\, b_A\, \Delta x_A \left(\frac{\lambda_A + 1}{2\, U_A}\right) \left(\frac{n_A + 1}{2}\right)^{1/2} = \pi\, b_B\, \Delta x_B \left(\frac{\lambda_B + 1}{2\, U_B}\right) \left(\frac{n_B + 1}{2}\right)^{1/2}, \tag{13}$$

where the subscripts A and B refer to the incident and transmitted shock side of the interface, respectively. This equation gives the criterion for matching the meshes on two sides of an interface. Given the values of n for the two materials and the speed and compression ratios of the two shocks, this equation determines the correct mesh for the transmitted shock side of the interafce if the errors caused by the transient shock are to be minimized. From a practical point of view it is unfortunate

that Eq. (13) involves the compression ratio and speed of the transmitted shock, but the validity of this formula, and the sensitivity of the matching as the parameters vary, has been investigated by some numerical experiments which are described in the next section.

In some circumstances there will also be a reflected shock which leaves behind several meshes near the interface at the wrong density. Although the procedure described here could correct the density profile in the second material it cannot be expected to affect the error caused by the reflected shock. However, this error will be smaller than for the transmitted shock since the entropy jump $\Delta S/S$ is smaller.

IV. Some Numerical Results

Various numerical tests have been carried out using a one-dimensional hydrodynamic finite difference computer code primarily to investigate the nature and magnitude of the errors occurring when a viscous shock crosses an interface and secondly to test the theory of the preceding section. The value of b in general use is about 1.6, and this spreads the shock over about three meshes. Much smaller values of b lead to large oscillations behind the shock and much larger values spread the shock out over many meshes with a lack of definition of the shock position. The inclusion of a von-Neumann viscosity term reduces the time step of most difference methods by a factor proportional to $1/b$, and this is a deterrent to taking a value of b larger than is really necessary to represent the shock satisfactorily. However, in order to magnify the errors introduced by artificial viscosity terms and to keep arbitrary oscillations to a minimum, a larger value of b (viz. $b = 8$) has been used for most of the runs described here.

Figures 1a–d show the density profiles at various times that result when a von-Neumann shock wave of $\frac{1}{3}$ Mb crosses a sudden change of mesh of 1–5 cm in aluminium: a constant value of $b = 8$ has been used. It can be seen that there is a small reflected perturbation which travels away from the interface at the local speed of sound. The dip at the interface in the density profile at 150 μsec is permanent.

Since there is no change of material or shock velocity at the interface, Eq. (13) reduces to

$$b_A \, \Delta x_A = b_B \, \Delta x_B, \tag{14}$$

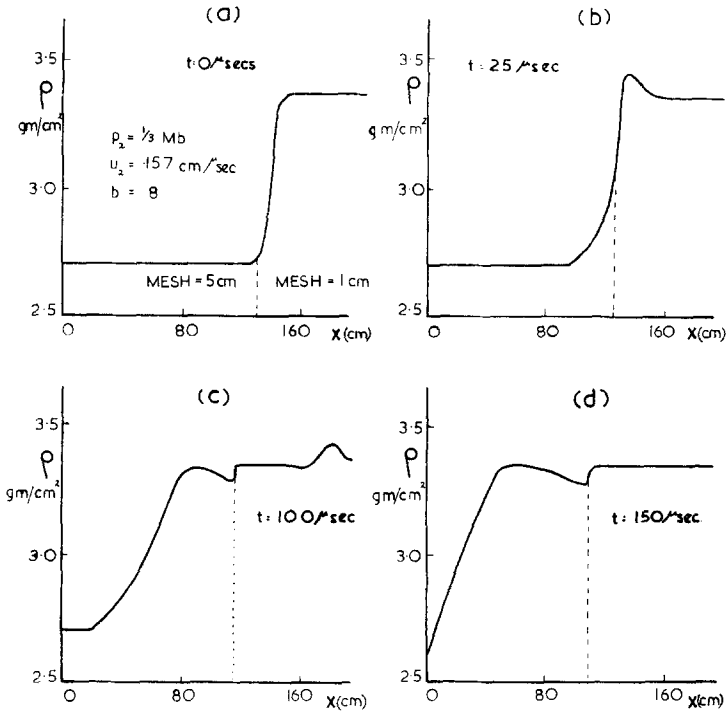

FIG. 1. Density profiles of a steady-state viscous shock ($p_2 = \frac{1}{3}$ Mb) as it crosses a change in mesh size. The value of b is kept constant.

and we can only match at the interface by altering b. In many cases this is not practical when $\Delta x_B/\Delta x_A$ is as large as 5; but, because our initial value of b is so large, we can reduce b to 1.6 in the second material without affecting the representation of the shock. Figure 2-a shows the results for such a run; it can be seen that the reflected perturbation and the "dip" at the interface wave been eliminated. The oscillations behind shock front arise from the reduced value of b.

Reducing b is often not practicable and an alternative is to change the mesh gradually rather than suddenly. Figure 2-b shows the resulting density profile of a run in which the mesh was changed from 1.0 to 5.0 cm in 8 steps. The amplitude of the reflected perturbation is reduced though its width is increased: a similar observation may be applied to the "dip" at the interface, although the net error in $\int \varrho dx$ is about the same in the two cases. We conclude that the continuous variation of the mesh does not eliminate the error but only spreads it out.

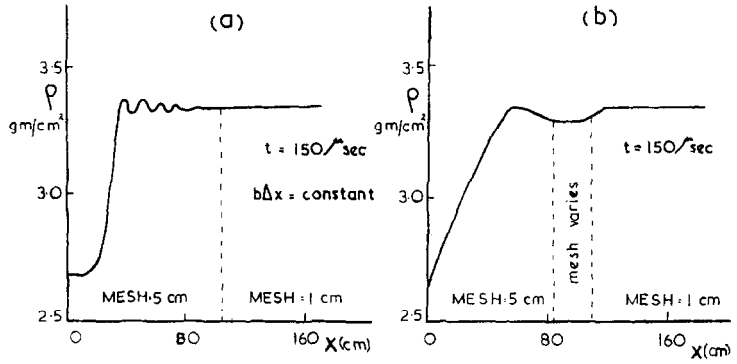

FIG. 2. Density profiles after a steady-state viscous shock ($p_2 = \frac{1}{3}$ Mb) has crossed a change in mesh size. (a) Abrupt change, but $b\,\Delta x = 8$ kept constant; (b) gradual change in mesh but b kept constant.

A series of runs has been carried out in which a shock crosses from aluminium to uranium where initially the materials are considered to be at their normal density: there will be a transmitted and reflected shock. Figures 3-a and 3-b show some typical nondimensional density profiles for two runs with the same mesh size but with two different values for b, viz. 1.6 and 8.0; it may be seen that the maximum error is about the same in both cases but that more meshes are affected for the larger value of b. This is what we would expect as the shock is spread out further for the larger value of b. Although the magnitude of the error at the interface is almost independent of b, it increases as the shock strength increases. A full quantitative investigation has not been carried out, but Fig. 3-a and 3-b show the density profile for the transmitted and reflected shock when the pressure driving the shock is 1 Mb. Figure 4 shows various density profiles most of which are referred to later but, for comparison with Figs. 3-a and 3-b, the continuous line in Fig. 4-a shows the density profile when the pressure driving the shock is $\frac{1}{3}$ Mb and $b = 8$.

A series of runs was carried out with $b = 8$, a shock pressure of $\frac{1}{3}$ Mb, and a constant mesh in the aluminium of 1 cm but with various mesh sizes in the uranium. The data for the shocks involved is

$$b = 8$$

Incident shock $\qquad X_A = 16.7\,\Delta x \qquad T_A = 21.2\,\Delta x \qquad q_{\max} = .034$

$$T_A/T_B = 0.58$$

Transmitted shock	$X_B = 14.1\ \Delta x$	$T_B = 36.6\ \Delta x$	$q_{\max} = .0805$	$T_A/T_C = 0.81$
Reflected shock	$X_C = 17.7\ \Delta x$	$T_C = 26.0\ \Delta x$	$q_{\max} = .0074$	

The value of 0.58 given here for T_A/T_B has been calculated using the theory of the previous section as given by Eq. (13); it tells us that the ratio of the mesh in the uranium to the mesh in the aluminium, ν, should be 0.58 in order to make $T_A/T_B = 1$ and reduce the density errors to a minimum.

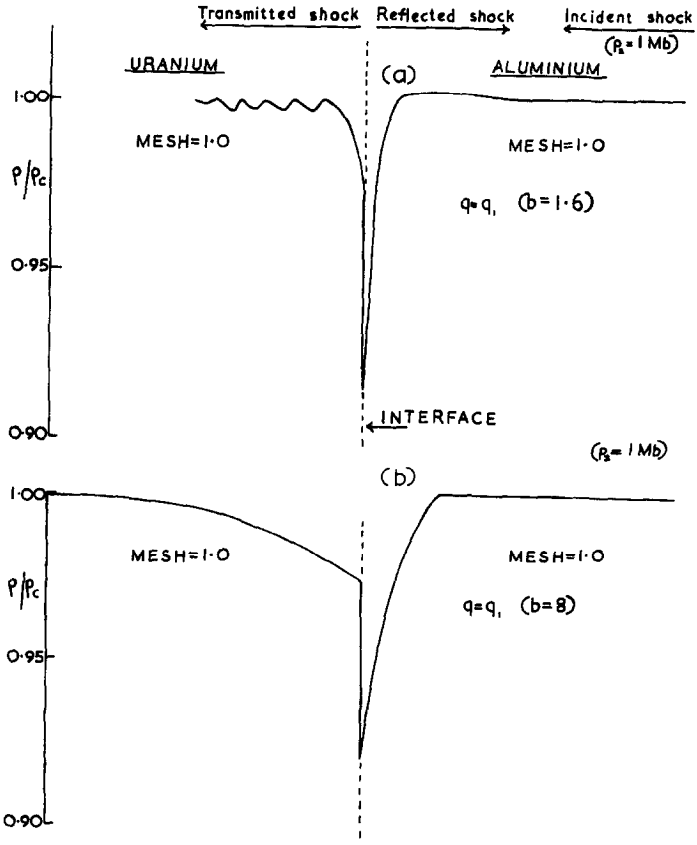

FIG. 3. Normalized density profiles after a steady-state viscous shock ($p_2 = 1$ Mb) has crossed an Al-U interface. (a) $b = 1.6$; (b) $b = 8$. ϱ_c is the correct value of the density for the steady-state shock in the particular material considered.

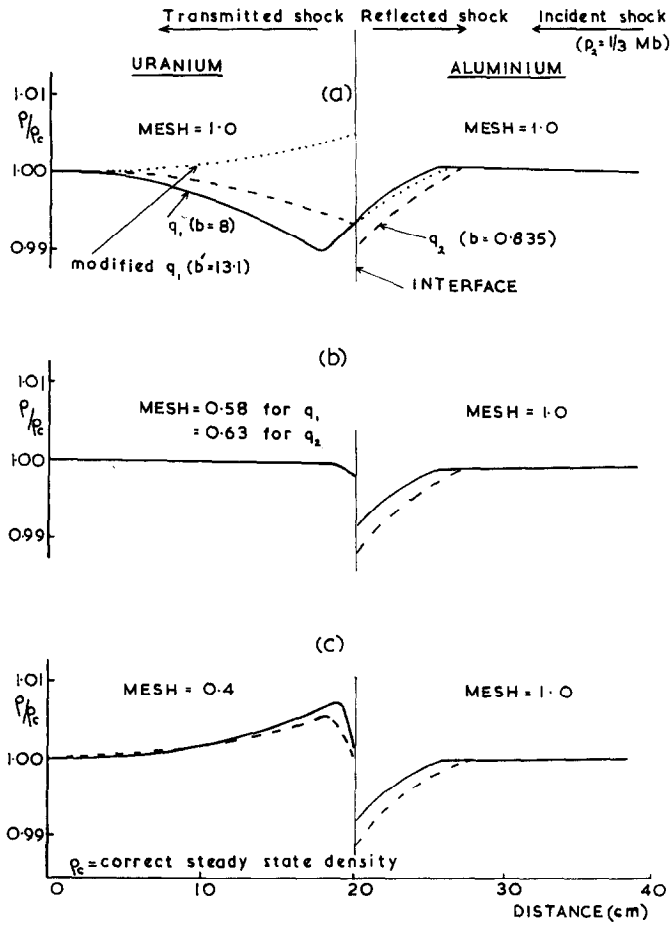

FIG. 4. Normalized density profiles after a steady-state viscous shock ($p_2 = \frac{1}{3}$ Mb) has crossed an Al-U interface. Continuous curves refer to the von-Neumann q term; large broken curves to the Landshoff q [Eq. (20)]. (a) Mesh unchanged; (b) optimum mesh ratio; (c) smaller mesh ratio than optimum.

The results of the runs are shown in Figs. 4–6. In Fig. 4 the continuous lines show the density profiles for three values of v for the von-Neumann q; the broken lines are profiles for different tyeps of q considered later. Figure 5 shows how rapidly the shock reaches its new state in the uranium for various values of v; the maximum q in the uranium is plotted against the distance moved by the interface. Theory indicates that v should be 0.58, and Fig. 5 shows that the transmitted shock settles down to its new steady state most rapidly for this value of v. The broken

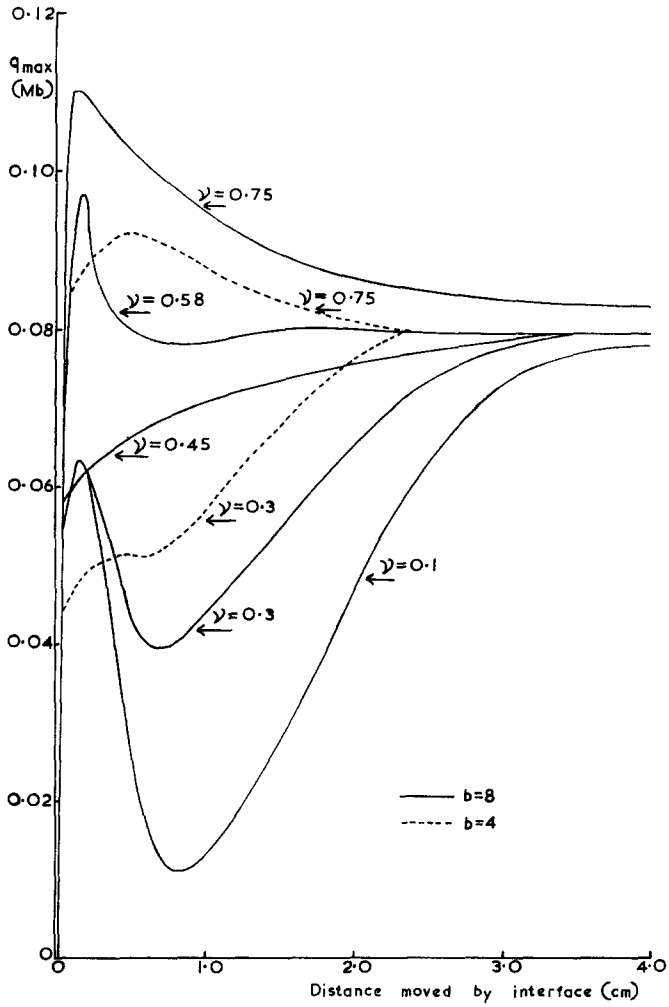

FIG. 5. Maxima of von-Neumann q for the transmitted shock after a steady-state viscous shock ($p_2 = \frac{1}{3}$ Mb) has crossed an Al-U interface.

lines in this figure show the effect of using a smaller value of b. The density profile for $v = 0.58$ shown in Fig. 4-b has only a small error in the uranium. For the shock driven by a pressure of 1 Mb considered earlier, v_c is 0.63; this is not significantly different from 0.58, and the increase in the error for the stronger shock is because the shocks are stronger and not because of any difference in the nature of the interaction at the interface.

It can be seen from Fig. 5 that a small variation in v_c increases the distance required for the shock to settle to its new state by quite a large factor, and this increase is naturally reflected in an increase in the number of cells at the wrong density. These results are shown in Figs. 6-a and 6-b, which emphasize just how good the matching needs to be to obtain the best possible result. Figure 6-b shows how far the shock

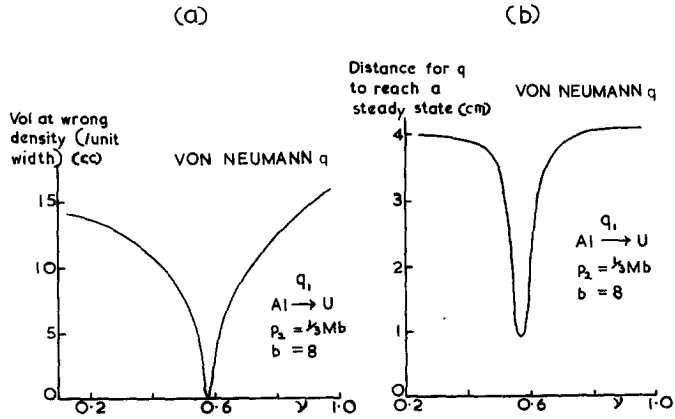

FIG. 6. Dependence on v of errors in the transmitted shock after a steady-state viscous shock ($p_2 = \frac{1}{3}$ Mb) has crossed from Al to U.

travels after first meeting the interface before it reaches its new steady state; we can see that once v varies from v_c by more than about 25% the distance required for the shock to reach its new state varies only slowly; the volume at the wrong density, however, continues to rise. Investigation of the results of this series of runs and others shows that the maximum error in the density increases linearly as v/v_c varies from 1.0, the exact rate depending on the shocks involved.

V. A VON-NEUMANN q WITH A VARIABLE b

We have seen how to correct the error caused by the transmitted shock, but this does not correct the error caused by the reflected shock. This error could be corrected if we introduce a variable value for the viscous constant b, i.e., if we write kb_0 for b, where

$$k = \frac{2}{\lambda + 1} U \left(\frac{n+1}{2}\right)^{1/2}, \tag{15}$$

then

$$X = \pi b_0 U \Delta x \tag{16}$$

and

$$T = \pi b_0 \Delta x, \tag{17}$$

an thus, without altering Δx, we can keep T constant throughout. The success of this method depends on finding the correct values of U and λ. This is easy for a single shock but not when multiple interactions and shocks are involved.

If we denote the constant states ahead and behind of the shock front by subscripts 1 and 2 we have from Eqs. (1) to (3) and Eq. (5) that

$$U = \left\{ \frac{1}{\varrho_1} \left[\frac{n+1}{2} p_2 + b + \frac{n-1}{2} p_1 \right] \right\}^{1/2}, \tag{18}$$

$$\lambda = \frac{\varrho_1}{\varrho_2} = \frac{n-1}{n+1} + \frac{2(b + np_1)}{(n+1)\left[\left(b + \frac{n-1}{2} p_1 \right) + \frac{n+1}{2} p_2 \right]}. \tag{19}$$

A simple way of approximating to U and λ at any point is to use Eqs. (18) and (19) where p_2, p_1, ϱ_2, and ϱ_1 are evaluated from the values several points shead and behind of the point being calculated. Figure 7

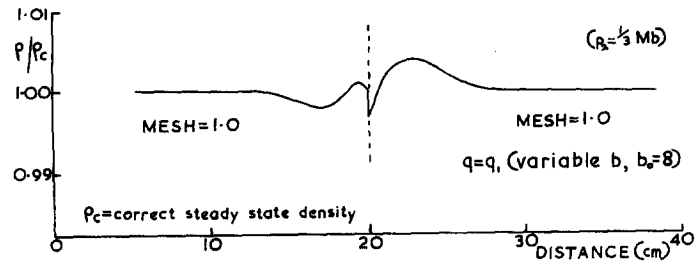

FIG. 7. Normalized density profile after a steady-state viscous shock ($p_2 = \frac{1}{3}$ Mb) with a variable value of b has crossed from Al to U.

shows the results of a run with such a q; they are not perfect but could probably be improved by using more accurate, but necessarily more complicated, methods of evaluating U and λ at the mesh points.

VI. The Landshoff q Term

Another form of q that is sometimes used in one that is linear instead of quadratic in $\partial u/\partial x$. This is a form first used by Landshoff [3], and in terms of the Eulerian derivative it may be written

$$q_2 = \begin{cases} L\,\varrho_0 c \left|\dfrac{\partial u}{\partial x}\right| & \dfrac{\partial u}{\partial x} < 0, \\ 0 & \dfrac{\partial u}{\partial x} > 0, \end{cases} \tag{20}$$

where c is the velocity of sound. With this form of q the shock is spread out over a distance X, where

$$X = L\left(\frac{2}{n+1}\right)^{1/2} \int_\lambda^1 \frac{y\left(\lambda - \dfrac{n-1}{n+1}y^2\right)^{1/2} dy}{(1-y)(y-\lambda)}. \tag{21}$$

As $y \to 1$ or λ it is obvious that this integral diverges giving an infinite shock width X; however, we can modify the limits of the integration to $\lambda + (1-\lambda)/100$ and $1 - (1-\lambda)/100$ to give a finite value for X. This modified definition of the shock width is found to give satisfactory results over the complete range of λ.

The integral in Eq. (21) cannot be evaluated in closed form and must be calculated numerically. This has been done for various values of n and λ and the results are shown in Fig. 8. If we have a strong shock and a perfect gas equation of state [$B = 0$ in Eq. (5)], then

$$\lambda = \frac{n-1}{n+1} \tag{22}$$

and X/L only depends on n or λ. This dependence is shown in Fig. 8 as the small broken curve labelled *strong "ideal" shock*. The values of X/L are nondimensional, but because of the different range of values it is not practical to show the comparison between X/L for the Landshoff and von-Neumann q in the same figure. However, Fig. 9 shows values of X/L for the von-Neumann q, and also a non-dimensional shock transit time T/L' for a modified von-Neumann q which is considered in the next section (L' is a modified definition of L). The values of X/L are calculated from Eq. (9) and are shown as the large broken line; as before, the small broken line shows the values of X/L for a strong

shock with a perfect gas equation of state. Comparing the two sets of curves it can be seen that X/L varies much more for the Landshoff q than for the von-Neumann q, especially as $\lambda \to 1$. Now for matching the mesh in two materials it has been shown that we would like the time T to remain constant, but for practical reasons concerning the definition of the shock it is also desirable that X should not vary too much. This is therefore a disadvantage of the Landshoff q.

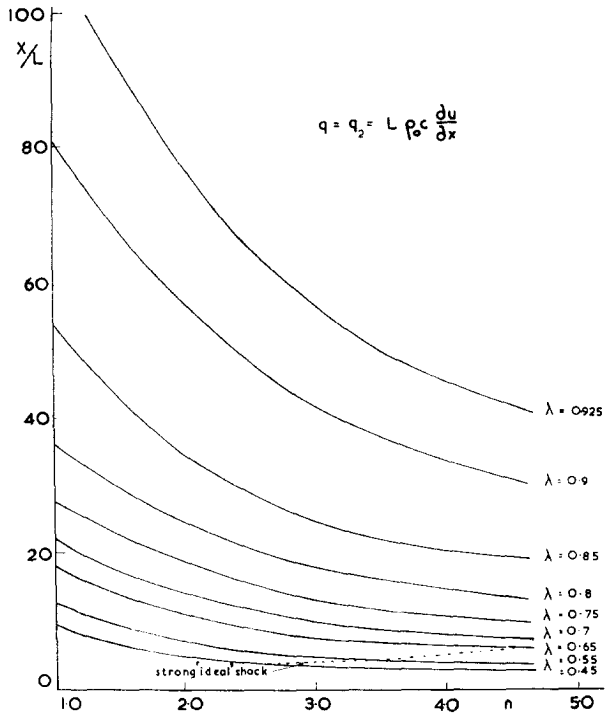

FIG. 8. Nondimensional shock width for the Landshoff (linear) q.

Using the shock widths shown in Fig. 8 it is possible to obtain the critical value of ν for a given shock and interface. We consider the same shock of $\frac{1}{3}$ Mb crossing from aluminium to uranium, and choose b such that the shock is spread over the same distance in the aluminium as the von-Neumann q spreads it out with $b = 8$. We then have the following data for the three shocks:

$$b = 0.835$$

Incident shock	$X_A = 16.7\ \Delta x$	$T_A = 24.8\ \Delta x$	$T_A/T_B = 0.63$
Transmitted shock	$X_B = 14.7\ \Delta x$	$T_B = 38.6\ \Delta x$	$T_A/T_C = 0.38$
Reflected shock	$X_C = 45.5\ \Delta x$	$T_C = 66.5\ \Delta x$	

The critical value of ν is therefore seen to be 0.63.

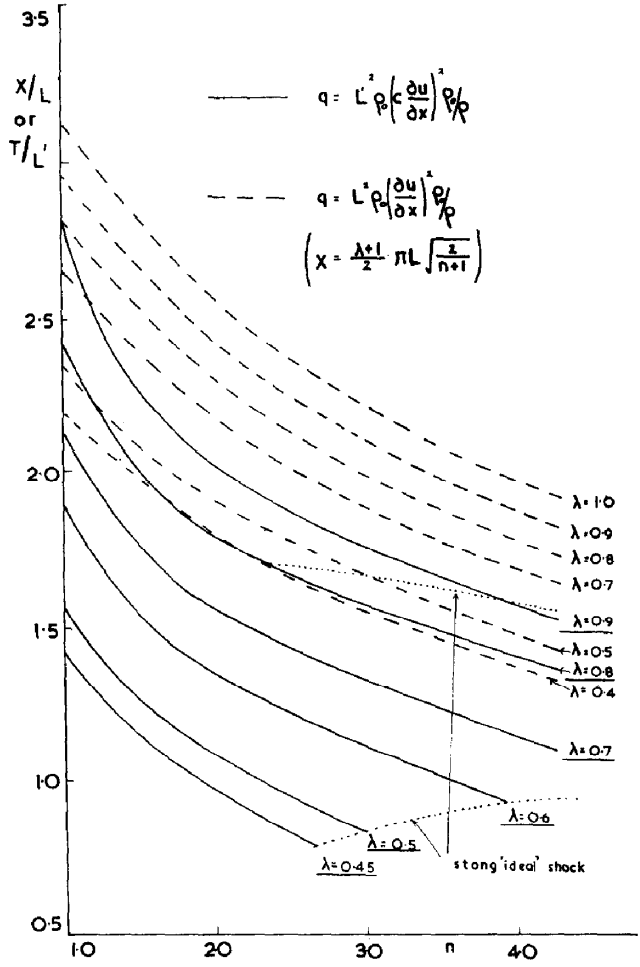

FIG. 9. Nondimensional shock width and shock transit time for the von-Neumann q and modified von-Neumann q, respectively.

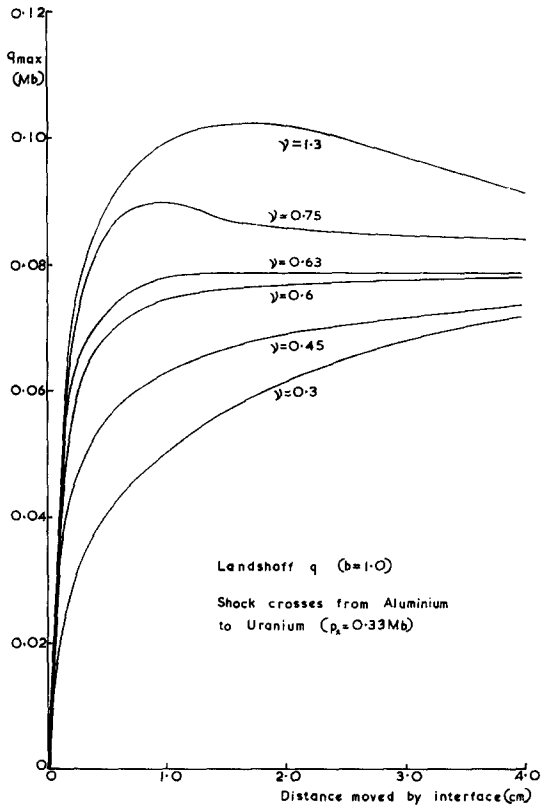

FIG. 10. Maxima of Landshoff q for the transmitted shock after a steady-state viscous shock ($p_2 = \frac{1}{3}$ Mb) has crossed an Al-U interface.

Figure 10 shows the distance required for the shock to reach its new steady state for various values of ν; it may be seen that the best curve is the one corresponding to $\nu = 0.63$; that this is the best choice of ν is confirmed by the density profiles. The curves are different from those obtained for the von-Neumann q in that, for $\nu > \nu_c$, instead of a rapid overshoot there is a steady rise to a maximum followed by a longish tail. This longish tail also exists for $\nu < \nu_c$. Some density profiles for q_2 are shown as the large broken lines in Fig. 4: for the same value of ν the errors do not seem quite so large as for q_1. Figure 11 shows the volume of material at the wrong density for this type of q, and comparing this figure with Fig. 6-1 it can be seen that the volume at the wrong density does not seem to be quite so sensitive to the value of ν for the Landshoff q as for the von-Neumann q.

It has not been found possible to reduce the density errors by taking linear combinations of the two forms of q (X naturally being kept constant). There is the possibility that the error could be reduced by taking a form of q proportional to $(\partial u/\partial x)^{1/2}$; however the shock profiles for such a q appear to have long tails, with a relatively rapid change near the center. This does not produce a very satisfactory profile, and although the trouble could probably be reduced by modifying the definition of

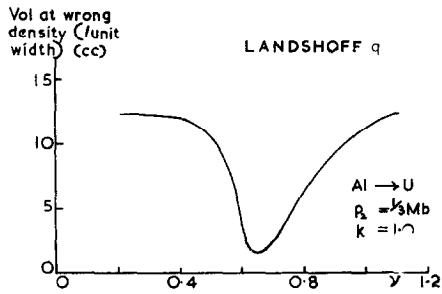

FIG. 11. Dependence on v of errors in the transmitted shock after a steady-state viscous shock ($p_2 = \frac{1}{3}$ Mb) has crossed from Al to U.

the shock width and cutting q off below a certain value, this form has not been pursued.

VII. Forms of q with an Extra Velocity of Sound Term

It has been shown that in order to eliminate the errors caused when a viscous shock crosses an interface, it is necessary to match T on both sides of the interface. If $T/b\ \Delta x$ was a constant independent of λ, n, and U, then matching would be straightforward, both for transmitted and reflected shocks without changing the mesh at an interface. A slight variation in T leads to a large increase in the number of meshes at the wrong density, although the maximum error is proportional to the error in T. Thus, although exact matching is the ideal, the better the match, the less the net error.

For practical examples, $0.6 < \lambda < 1$ and $1 < n < 4$; therefore $0.8 < (\lambda + 1)/2 < 1.0$, and $0.6 < [2/(n + 1)]^{1/2} < 1$: these changes are much less than possible variations in U, which can fluctuate widely from material to material. It is therefore plausible that for a constant

mesh the net error could be reduced if the definition of q was modified to include the shock speed U. For the general case, however, this is not practical and a compromise is to introduce the sound speed, c, which is closely related to U by

$$c = U\left(\frac{n+1}{2}\right)^{1/2}\left(\lambda - \frac{n-1}{n+1}y^2\right)^{1/2}. \tag{23}$$

This formula assumes that the material ahead of the shock is at rest; if this is not so U is replaced by w_1, the velocity relative to the shock front of the material ahead of the shock. We therefore define a q which, in terms of the Eulerian derivative, may be written as

$$q_3 = \frac{\varrho_0^2}{\varrho}(b'\Delta x)^2 c^2\left(\frac{\partial u}{\partial x}\right)^2; \tag{24}$$

b' is still a constant but with the dimensions of (1/velocity). With this definition we have

$$T/L' = \int_\lambda^1 \frac{y\left(\lambda - \dfrac{n-1}{n+1}y^2\right)^{1/2}}{[(1-y)(y-\lambda)]^{1/2}}\,dy, \tag{25}$$

where $L' = b'\,\Delta x$. The integral may be evaluated, and the large broken lines in Fig. 9 show values of T/L' as a function λ and n: the small broken line again represents the curve obtained for a perfect gas and strong shocks. Given λ and n for the incident and transmitted shocks this figure shows directly how good or bad the matching will be if the same mesh is used for both materials; alternatively, of course, it shows what the correct value of ν should be for perfect matching. For very small values of n, T rises rapidly, and for a constant mesh we would therefore expect poor matching when one of the materials is characterized by a low value of n.

For the example considered earlier, again keeping $X/\Delta x$ constant at 16.7 in the aluminium, we have

$$b = 13.1 \qquad T_A/T_B = 1.17 \qquad \text{and} \qquad T_A/T_C = 0.87.$$

To compare this q with q_1 and q_2, the appropriate density profile for $\nu = 1$ is shown as the dotted line labelled "modified q_1" in Fig. 4-a. As would be expected from the value of T_A/T_C the error in the uranium has been considerably reduced, although the error in the aluminium is

about the same in spite of the improvement to T_A/T_C. Of course, as ν varies from 1.0 the errors will increase as before.

The improvement obtained by the use of q_3 instead of q_1 arises from the fact that the variation of shock width X is increased by the use of the modified form. This is satisfactory provided that we do not violate the practical conditions inherent in the use of an artificial viscosity, that is, X must be greater than about 3 meshes if the oscillations behind the shock are to be small and not so large that details of the flow pattern are lost. Although it depends on the type of problem being solved, changes in the shock width of the order two, as arise for this type of viscosity term, are probably satisfactory provided that b is chosen sufficiently large enough to represent all the shocks correctly.

For strong shocks and an ideal equation of state it may be noted that T/L' is just a function of n; thus, if this function is included in the definition of q, T/L' is a constant under these special conditions.

There is an analogous modification to the Landshoff q for which we may derive the following data:

$$b = 1.2 \qquad T_A/T_B = 1.37 \qquad \text{and} \qquad T_A/T_C = 0.4.$$

Compared with the ordinary Landshoff q there is an improvement in the density profile in the uranium, though not such a significant one as was obtained by modifying the von-Neumann q: this is consistent with the values of T_A/T_B obtained. The values of T/L' for this form of q are very similar to the values of X/L obtained for the ordinary Landshoff q and have not been shown here. Because of the large variations in T and the even larger variations in X, the modified Landshoff q is probably not a satisfactory form of q.

VIII. Conclusions

From the results and graphs obtained in the preceding sections we may draw the following conclusions about the effect of using viscous shocks in finite difference methods. When a viscous shock crosses an interface the density of the material close to the interface on the transmitted shock side will be left in error unless the ratio of the meshes on either side of the interface has a certain value ν_c which depends on the structure of the shock and its speed. If $\nu > \nu_c$ the density of this material

will be too low and vice versa. As v/v_c varies from 1.0 the number of cells at the wrong density increases rapidly, while the maximum error in density increases linearly. If there is also a reflected shock, then the final density of the material on the incident shock side of the interface will also be incorrect. This error can only be corrected by introducing a variable viscous constant, although the use of this method is limited since the values of b must be within a certain range.

At a change in material it would appear that the choice of mesh is not so critical for a Landshoff q as it is for a von-Neumann q. However, against this must be weighed the disadvantage of an increased shock width for values of λ close to 1.0. The modified Landshoff q term is not to be recommended because of large variations in the shock width: the modified von-Neumann q is more satisfactory from this point of view and, provided the moderate variations in shock width can be tolerated, has proved to be the best simple form of q tried.

Acknowledgments

The author would like to thank Dr. H. H. M. Pike and Dr. N. E. Hoskin for helpful discussions in the course of the work described in this paper.

References

1. J. von-Neumann and R. D. Richtmyer, *J. Appl. Phys.* **21**, 232–237 (1950).
2. L. Fox, *in* "Numerical Solution of Ordinary and Partial Differential Equations," Chapter 28. Pergammon Press, London (1962).
3. R. Landshoff, Report No. 1930. Los Alamos Scientific Laboratory (1955).